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Investigation to analytic solutions of modified conformable time–space fractional mixed partial differential equations

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ABSTRACT

In this paper, the invariant subspace method (ISM) is developed to obtain the exact solution of linear and nonlinear time and space fractional mixed partial differential equations involving modified conformable fractional derivative (MCFD). Moreover, the method is successfully extended to the coupled system of modified conformable fractional differential equations. Variety of time-space fractional mixed PDEs are considered and solved to illustrate the established result of ISM. Finally, a comparison between exact solution of fractional PDE in the sense of MCFD and Caputo fractional derivative (Ca-FD) is presented through some examples of time–space fractional mixed partial differential equations.

1. Introduction

Fractional calculus has several applications in science and engineering.[1–4] It is extensively used to study the models of physical and engineering phenomena in the form of fractional partial differential equations.[5–7] A lot of work has been reported on existence and uniqueness of solutions of FDEs (see Refs. 8, 9 and references therein). Several definitions of the fractional derivative have been introduced in the literature, such as the Riemann–Liouville definition,[4] the Caputo definition,[4] the Hilfer fractional definition,[10] the Caputo–Fabrizio fractional definition[11] etc. In 2014, Khalil et al.[12] gave a new definition of fractional derivative which is called conformable fractional derivative(CFD) and it is shown that "the computation of fractional derivative using this definition is easier than the other available definitions". The CFD of order α is defined for a function $f : [0, \infty) \to \mathbb{R}$ by Ref. 13

$$T^{\alpha} f(t) = f^{\alpha}(t) = \lim_{\epsilon \to 0} \frac{f^{\lceil \alpha \rceil - 1}(t + \epsilon (t)^{\lceil \alpha \rceil - \alpha}) - f^{\lceil \alpha \rceil - 1}(t)}{\epsilon}, \tag{1.1}$$

where $n \in \mathbb{N}$, $n - 1 < \alpha \leq n$, $t > 0$ and $\lceil \alpha \rceil$ is the smallest integer number greater than or equal to α. Note that $T^{\alpha} f(0) = \lim_{t \to 0^+} T^{\alpha} f(t)$, provided limit exists and $f(t)$ is n-differentiable.

Recently, many research papers have been published concerning the results on CFD (see Refs. 12–14, and references therein). In Ref. 15, authors made use of residual power series method to present the exact solution of time-fractional Schrödinger equation (TFSE) with conformable derivative and provided a graphical comparison to indicate that CFD is a suitable alternative to Caputo fractional derivative in the modelling of TFSE.

The main advantage of the CFD is that it satisfies the concepts and rules of an ordinary derivative such as: quotient, product and chain rules while the other fractional definitions fail to meet these rules. Moreover, it can be efficiently extended to solve the fractional differential equations and systems exactly and numerically. Despite of these, there are some properties which are not followed by the CFD. For example, (i) As most definitions of fractional derivatives, the CFD does not satisfy commutative property and (ii) the sequential derivatives do not coincide with the higher order derivatives. To fulfil these properties, in (2020), Ahmad El-Ajou[16] did modification in the CFD and presented a definition known as modified conformable fractional derivative (MCFD). The modified conformable fractional derivative is now attracting the researchers due to its various nice properties. Some basic properties of the MCFD are given in the preliminaries.

Exact solutions play a vital role in the proper understanding of qualitative features of the concerned phenomena and processes in various areas of science and engineering. Numerical methods such as Adomian decomposition method,[17] new iterative method (NIM),[18] homotopy perturbation method[19] have been employed to solve FDEs. However, solutions obtained by all these methods are local in nature and it is important to explore other techniques to find the exact analytical solutions of FDEs. The invariant subspace method (ISM) is a very effective method which can be used for obtaining the exact solutions of fractional partial differential equations. It is widely used to obtain the exact solution of fractional differential equations with Riemann–Liouville and Caputo fractional derivatives.[20–22] Recently (2020), K. Hosseini et al.[23] successfully utilized ISM to get the exact solution for integer order nonlinear water wave equation(NWWE) and nonlinear

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dispersive water wave equations. In Ref. 14, M. S. Hashemi employed ISM to the time fractional differential equations with CFD.

In this paper, we develop the invariant subspace method for finding the exact solution to nonlinear modified conformable fractional-order mixed partial differential equations(including both fractional space and time derivatives). The equation reads as:

$$\sum_{i=1}^{m} a_i\, {}_{t_0}T_t^{i\alpha} u(x,t) - a\, {}_{t_0}T_t^{\gamma}\, {}_{x_0}T_x^{\delta} u(x,t) =$$

$$F(x, u, {}_{x_0}T_x^{\beta} u(x,t), \dots, {}_{x_0}T_x^{n\beta} u(x,t)) = F(u), \tag{1.2}$$

where $\alpha \in (k_1 - 1, k_1]$ and $\beta \in (k_2 - 1, k_2]$ i.e. $\alpha \in \chi_\alpha$ and $\beta \in \chi_\beta$ for $\delta \in (0,1]$, $\gamma \in (0,1]$, $k, k_1, k_2 \in N$; $a, a_i \in R$. $F(u)$ is linear/nonlinear operator of u and its fractional derivatives.

Also, we develop the invariant subspace method for finding the exact solutions to coupled time-space fractional mixed modified conformable partial differential equations:

$$\sum_{j=1}^{m_1} a_j\, {}_{t_0}T_t^{j\alpha} u(x,t) - \mu_1\, {}_{t_0}T_t^{\gamma_1}\, {}_{x_0}T_x^{\delta} u(x,t) =$$

$$F_1(x, u, v, {}_{x_0}T_x^{\beta} u(x,t), \dots, {}_{x_0}T_x^{l_1\beta} u(x,t), {}_{x_0}T_x^{l_2\beta} v(x,t)) = F_1(u,v), \tag{1.3}$$

$$\sum_{k=1}^{m_2} c_k\, {}_{t_0}T_t^{k\alpha} v(x,t) - \mu_2\, {}_{t_0}T_t^{\gamma_2}\, {}_{x_0}T_x^{\delta} v(x,t) =$$

$$F_2(x, u, v, {}_{x_0}T_x^{\beta} u(x,t), \dots, {}_{x_0}T_x^{l_3\beta} u(x,t), {}_{x_0}T_x^{l_4\beta} v(x,t)) = F_2(u,v), \tag{1.4}$$

where $\alpha \in (k_1 - 1, k_1]$ and $\beta \in (k_2 - 1, k_2]$, i.e. $\alpha \in \chi_\alpha$ and $\beta \in \chi_\beta$ for $\delta \in (0,1]$, $\gamma \in (0,1]$, $k_1, k_2, l_1, l_2, l_3, l_4 \in N$; $\mu_1, \mu_2, a_j, c_k \in R$. $F_1(u,v)$ and $F_2(u,v)$ are linear/nonlinear operators of u, v and their fractional derivatives with respect to the independent variable x.

The layout of this paper is as follows. In Section 2, we present some definitions and important properties of modified conformable fractional derivatives. In Section 3, the invariant subspace method is established for the modified conformable time-space fractional mixed partial differential equations. In Section 4, various examples of nonlinear fractional PDEs (including initial value problems) are given to illustrate the developed method. Finally, Section 5 provides concluding remarks.

2. Preliminaries and notations

This section recalls basic definitions, notations and preliminary results which will be used throughout the paper. We use the standard notations, R, N for denoting the set of real numbers and natural numbers, respectively. Traditionally, $C[s,\infty)$ and $C(-\infty,s]$ represent the set of real valued continuous functions on $[s,\infty)$ and $(-\infty,s]$, respectively. $f^{n\alpha}(t)$ represents CFD of the function f of order $n\alpha$.

Definition 2.1 (*Caputo Fractional Derivative (Ca-FD)*). Caputo fractional derivative of order $\alpha > 0$ of $f \in C^n([a,b],R)$ is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^n(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau,$$

where $n-1 < \alpha < n$ and for $\alpha = n$, we obtain usual nth order derivative $f^n(t)$.

Modified conformable fractional derivative.

Definition 2.2 (*See Ref. 16*). Let $(\alpha : 0 < \alpha \le 1)$ be a cyclic subgroup of $(R,+)$. The number β is said to be a class of α iff $\beta \in (\alpha : 0 < \alpha \le 1)$. Note that if $\beta \in (n-1, n]$ for $n \in N$, then $\beta \in (\alpha : \frac{n-1}{n} < \alpha \le 1, \alpha = \frac{\beta}{n}) = \chi_\alpha$ and it is the main idea of the definition of MCFD.

Remark 2.1. For $\beta \in (n-1, n]$, there exists a unique $\alpha \in (0,1]$ such that $\beta \in \chi_\alpha$ and $\beta = n\alpha$. For example, $\beta = 1.51$ belongs to the class of $\chi_{0.755}$.

Definition 2.3. The left-sided MCFD of order $\beta \in \chi_\alpha$ (where $\beta \in (n-1, n]$, $n \in N$) of a function $f : [s,\infty) \to R$ is defined by (See Ref. 16):

$$\begin{aligned} T_s^\beta f(t) &= \lim_{\epsilon \to 0} \frac{f^{((n-1)\alpha)}(t + \epsilon(t-s)^{1-\alpha}) - f^{((n-1)\alpha)}(t)}{\epsilon}, t > s, \\ T_s^\beta f(s) &= \lim_{t \to s^+} T_s^\beta f(t), \end{aligned} \tag{2.1}$$

provided the limit exists and $f(t)$ is $(n-1)\alpha$-left-sided-differentiable for $t > s$.

The right-sided MCFD of order $\beta \in \chi_\alpha$ (where $\beta \in (n-1, n]$, $n \in N$) of a function $f : (-\infty, s] \to R$ is defined by (See Ref. 16):

$$\begin{aligned} {}^\beta T_s f(t) &= (-1)^n \lim_{\epsilon \to 0} \frac{f^{((n-1)\alpha)}(t + \epsilon(s-t)^{1-\alpha}) - f^{((n-1)\alpha)}(t)}{\epsilon}, s > t, \\ {}^\beta T_s f(s) &= \lim_{t \to s^-} {}^\beta T_s f(t), \end{aligned} \tag{2.2}$$

provided the limit exists and $f(t)$ is $(n-1)\alpha$-right-sided-differentiable for $t < s$.

Remark 2.2. (1) If $\beta \in (0,1)$, then the modified conformable fractional derivative coincide with conformable fractional derivative and if $\beta \in N$, then we obtain the usual higher order derivatives.

(2) Every real number β such that $n-1 < \beta \le n$, belongs to an unique class of α which is χ_α.

(3) The left-sided sequential MCFD of order $\beta \in \chi_\alpha$ of a function $f : [s,\infty) \to R$ is defined by(See Ref. 16):

$$\begin{aligned} T_s^{m\beta} f(t) &= \lim_{\epsilon \to 0} \frac{f^{((mn-1)\alpha)}(t + \epsilon(t-s)^{1-\alpha}) - f^{((mn-1)\alpha)}(t)}{\epsilon}, t > s, \\ T_s^{m\beta} f(s) &= \lim_{t \to s^+} T_s^{m\beta} f(t), \end{aligned} \tag{2.3}$$

provided the limit exists and $f(t)$ is $(mn-1)\alpha$-left-sided-differentiable for $t > s$.

The right-sided sequential MCFD of order $\beta \in \chi_\alpha$ of a function $f : (-\infty, s] \to R$ is defined by (See Ref. 16):

$$\begin{aligned} {}^{m\beta} T_s f(t) &= (-1)^{mn} \lim_{\epsilon \to 0} \frac{f^{((mn-1)\alpha)}(t + \epsilon(s-t)^{1-\alpha}) - f^{((mn-1)\alpha)}(t)}{\epsilon}, s > t, \\ {}^{m\beta} T_s f(s) &= \lim_{t \to s^-} T_s^{m\beta} f(t), \end{aligned} \tag{2.4}$$

provided the limit exists and $f(t)$ is $(mn-1)\alpha$-right-sided-differentiable for $s > t$.

Lemma 2.1 (*See Ref. 16*). *Let $\beta \in \chi_\alpha$ and $\gamma \in R$. Then,*
(1) $T_s^\beta f(t) = (t-s)^{1-\alpha} \frac{d}{dt}(T_s^{(n-1)\alpha} f(t))$,
(2) ${}^\beta T_s f(t) = (-1)^n (s-t)^{1-\alpha} \frac{d}{dt}({}^{(n-1)\alpha} T_s f(t))$,
(3) $T_s^\beta(C) = {}^\beta T_s(C) = 0$, C is constant,
(4) $T_s^\beta (t-s)^\delta = \prod_{k=0}^{n-1} (\delta - k\alpha)(t-s)^{\delta-\beta}$.

Lemma 2.2 (*See Ref. 16*). *Let $\beta, \gamma \in \chi_\alpha$ and $t > s$. Then,*
(1) $T_s^\beta T_s^\gamma = T_s^{\beta+\gamma} = T_s^\gamma T_s^\beta$,
(2) ${}^\beta T_s {}^\gamma T_s = {}^{\beta+\gamma} T_s = {}^\gamma T_s {}^\beta T_s$,
(3) $T_s^{m\beta} = T_s^\beta T_s^\beta \dots T_s^\beta (m - times) = T_s^\alpha T_s^\alpha \dots T_s^\alpha (mn - times)$,
(4) ${}^{m\beta} T_s = {}^\beta T_s {}^\beta T_s \dots {}^\beta T_s (m - times) = {}^\alpha T_s {}^\alpha T_s \dots {}^\alpha T_s (mn - times)$.

Modified conformable fractional integral.

Definition 2.4 (*See Ref. 16*). (1) A real function $f(t)$, $t > s$ is said to be in the space C_μ, $\mu \in R$ if there exists a real number $k > \mu$ such that $f(t) = (t-s)^k f_1(t)$, for $f_1(t) \in C[s,\infty)$.

(2) A real function $f(t)$, $t < s$ is said to be in the space C_δ, $\delta \in R$ if there exists a real number $k > \delta$ such that $f(t) = (s-t)^k f_1(t)$, for $f_1(t) \in C(-\infty, s]$.

Definition 2.5. The left sided Modified conformable fractional integral (MCFI) of order $\beta \in \mathbb{Z}_\alpha$ of a function $f(t) \in C_{a}$, $\beta \geq -\alpha$, $t > a$ is defined by (See Ref. 16):

$$\mathfrak{I}^{\beta}_{a} f(t) = \frac{1}{\alpha^{m}\Gamma(m-1)} \int_a^t \frac{((t-a)^{\alpha} - (s-a)^{\alpha})^{m-1} f(s)}{(s-a)^{1-\alpha}} ds$$

The right sided Modified conformable fractional integral (MCFI) of order $\beta \in \mathbb{Z}_\alpha$ of a function $f(t) \in C_b$, $\beta \geq -\alpha$, $t < b$ is defined by (See Ref. 16):

$${}^{\beta}\mathfrak{I}_{b} f(t) = \frac{1}{\alpha^{m}\Gamma(m-1)} \int_t^b \frac{((b-t)^{\alpha} - (b-s)^{\alpha})^{m-1} f(s)}{(b-s)^{1-\alpha}} ds$$

Lemma 2.3 *(See Ref. 16).* *Let* $\beta, \mu \in \mathbb{Z}_\alpha$ *and* $t > a$. *Then,*
(1) $\mathfrak{I}^{\beta}_{a}\mathfrak{I}^{\mu}_{a} = \mathfrak{I}^{\beta+\mu}_{a} = \mathfrak{I}^{\mu}_{a}\mathfrak{I}^{\beta}_{a}$,
(2) ${}^{\beta}\mathfrak{I}_{b}{}^{\mu}\mathfrak{I}_{b} = {}^{\beta+\mu}\mathfrak{I}_{b} = {}^{\mu}\mathfrak{I}_{b}{}^{\beta}\mathfrak{I}_{b}$.

Lemma 2.4 *(See Ref. 16).* *If* $\alpha \in (0,1]$, $n \in \mathbb{N}$, $f \in C_{\mu}$, $\mu \geq -\alpha$ *and* $t > a$. *Then,*
(1) $\mathbb{T}^{n\alpha}_{a} f(t) \mathfrak{I}^{n\alpha}_{a} f(t) = f(t)$,
(2) $\mathfrak{I}^{n\alpha}_{a}\mathbb{T}^{n\alpha}_{a} f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k\alpha)}(a)\frac{(t-a)^{k\alpha}}{\alpha^k k!}$.
If $\beta \in \mathbb{Z}_\alpha$, $n \in \mathbb{N}$, $f \in C_\mu$ *and* $\mu \geq -\alpha$. *Then,*
(3) $\mathbb{T}^{n\beta}_{a} f(t)\mathfrak{I}^{n\beta}_{a} f(t) = f(t)$,
(4) $\mathfrak{I}^{n\beta}_{a}\mathbb{T}^{n\beta}_{a} f(t) = f(t) - \sum_{k=0}^{mn-1} f^{(k\alpha)}(a)\frac{(t-a)^{k\alpha}}{\alpha^k k!}$.

Invariant subspace.

Definition 2.6. Finite dimensional linear space $I_{n+1} = span\{I_0(x), I_1(x), \ldots, I_n(x)\}$ is said to be an invariant subspace with respect to the operator (linear/nonlinear) F if $F(I_{n+1}) \subseteq I_{n+1}$, where $I_i(x)$, $0 \leq i \leq n$ are continuous linearly independent functions of x.

Definition 2.7. Finite dimensional linear spaces $I_{n_1} \times I_{n_2} = span\{I^1_1(x), \ldots, I^1_{n_1}(x)\} \times span\{I^2_1(x), \ldots, I^2_{n_2}(x)\}$ is said to be an invariant subspace with respect to the operators (linear/nonlinear) F_1 and F_2 if $F_1(I_{n_1} \times I_{n_2}) \subseteq I_{n_1} \times I_{n_2}$ and $F_2(I_{n_1} \times I_{n_2}) \subseteq I_{n_1} \times I_{n_2}$, where $I^m_i(x)$, $1 \leq i \leq n_m$, $m = 1, 2$, are continuous linearly independent functions of x.

3. Invariant subspace method

In this section, we develop the invariant subspace method to the modified conformable time-space fractional mixed partial differential equations and coupled modified conformable time-space fractional mixed partial differential equations.

3.1. *Modified conformable time-space fractional mixed partial differential equations*

Consider the following modified conformable time-space fractional mixed partial differential equation

$$\sum_{j=1}^{m} a_j\, {}_t\mathbb{T}^{j\alpha}_0 u(x,t) - \mu\, {}_t\mathbb{T}^{\alpha}_0 ({}_x\mathbb{T}^{\beta}_0 u(x,t)) = F(u)$$
$$= F(x, u, {}_x\mathbb{T}^{\beta}_0 u(x,t), \ldots, {}_x\mathbb{T}^{k\beta}_0 u(x,t)). \tag{3.1}$$

where ${}_t\mathbb{T}^{j\alpha}_0 u(x,t)$, $j = 1, \ldots, m$ and ${}_x\mathbb{T}^{i\beta}_0 u(x,t)$, $i = 1, \ldots, k$ are the modified conformable time and space derivatives, respectively, $F(u)$ is a linear/nonlinear operator of u and its fractional derivatives, $\alpha \in (k_1 - 1, k_1]$ and $\beta \in (k_2 - 1, k_2]$, $k, k_1, k_2 \in \mathbb{N}$, i.e. $\alpha \in \mathbb{Z}_\delta$ and $\beta \in \mathbb{Z}_\gamma$ for $\delta \in (0, 1]$, $\gamma \in (0, 1]$, $a_j, \mu \in \mathbb{R}$.

Theorem 3.1. *Let a finite dimensional vector space* $I_n = span\{I_1(x), \ldots, I_n(x)\}$ *be the invariant subspace under the operator* $F(u)$ *and* ${}_x\mathbb{T}^{\beta}_0 u(x,t)$. *Then, FPDE (3.1) has an exact solution as*

$$u(x,t) = \sum_{i=1}^{n} \lambda_i(t) I_i(x), \tag{3.2}$$

where $\lambda_i(t)$ *for* $1 \leq i \leq n$ *satisfy a system of modified conformable fractional ODEs*

$$\sum_{j=1}^{m} a_j\, {}_t\mathbb{T}^{j\alpha}_0 (\lambda_i(t)) - \mu\, {}_t\mathbb{T}^{\alpha}_0 \phi_{n+i}(\lambda_1(t), \ldots, \lambda_n(t)) =$$
$$\phi_i(\lambda_1(t), \ldots, \lambda_n(t)), \; i = 1, \ldots, n \tag{3.3}$$

where $\phi_1, \ldots, \phi_n$ *are the expansion coefficients of* $F(u)$ *with respect to* I_n, $\phi_{n+1}, \ldots, \phi_{2n}$ *are the expansion coefficients of* ${}_x\mathbb{T}^{\beta}_0 u(x,t)$ *with respect to* I_n.

Proof. From $u(x,t) = \sum_{i=1}^{n} \lambda_i(t) I_i(x)$ and linearity of modified conformable fractional derivative, we obtain

$$\sum_{j=1}^{m} a_j\, {}_t\mathbb{T}^{j\alpha}_0 u(x,t) = \sum_{j=1}^{m} a_j\, {}_t\mathbb{T}^{j\alpha}_0 \left(\sum_{i=1}^{n} \lambda_i(t) I_i(x)\right) = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_j\, {}_t\mathbb{T}^{j\alpha}_0 \lambda_i(t)\right) I_i(x) \tag{3.4}$$

Since $I_n = span\{I_1(x), \ldots, I_n(x)\}$ is an invariant subspace under the operator $F(u)$ and ${}_x\mathbb{T}^{\beta}_0 u(x,t)$, there exist the expansion coefficient functions $\phi_1, \ldots, \phi_{2n}$ such that

$$F\left(\sum_{i=1}^{n} \lambda_i(t) I_i(x)\right) = \sum_{i=1}^{n} \phi_i(\lambda_1(t), \ldots, \lambda_n(t)) I_i(x), \tag{3.5}$$

$$\mu\, {}_x\mathbb{T}^{\beta}_0 \left(\sum_{i=n+1}^{2n} \lambda_i(t) I_i(x)\right) = \sum_{i=n+1}^{2n} \mu\, \phi_i(\lambda_1(t), \ldots, \lambda_n(t)) I_i(x), \tag{3.6}$$

where $\phi_1, \ldots, \phi_n$ are the expansion coefficients of $F(u)$ with respect to I_n, $\phi_{n+1}, \ldots, \phi_{2n}$ are the expansion coefficients of ${}_x\mathbb{T}^{\beta}_0 u(x,t)$ with respect to I_n.

From the Eqs. (3.5) and (3.6)

$$F(u(x,t)) + \mu\, {}_t\mathbb{T}^{\alpha}_0 ({}_x\mathbb{T}^{\beta}_0 u(x,t)) = \sum_{i=1}^{n} (\phi_i(\lambda_1(t), \ldots, \lambda_n(t)) I_i(x)$$
$$+ \mu\, {}_t\mathbb{T}^{\alpha}_0 \phi_{n+i}(\lambda_1(t) \ldots \lambda_n(t)) I_i(x)) \tag{3.7}$$

Eqs. (3.4) and (3.7) are substituted into Eq. (3.1) to obtain

$$\sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_j\, {}_t\mathbb{T}^{j\alpha}_0 \lambda_i(t) - \phi_i(\lambda_1(t), \ldots, \lambda_n(t)) - \mu\, {}_t\mathbb{T}^{\alpha}_0 \phi_{n+i}(\lambda_1(t) \ldots \lambda_n(t))\right) I_i(x) = 0 \tag{3.8}$$

Using Eq. (3.8) and the fact that $I_1(x), \ldots, I_n(x)$ are linearly independent, we have the system of FDEs as follows:

$$\sum_{j=1}^{m} a_j\, {}_t\mathbb{T}^{j\alpha}_0 \lambda_i(t) - \mu\, {}_t\mathbb{T}^{\alpha}_0 \phi_{n+i}(\lambda_1(t) \ldots \lambda_n(t)) = \phi_i(\lambda_1(t), \ldots, \lambda_n(t)), \tag{3.9}$$

where $i = 1, \ldots, n$.

Hence proved.

3.2. *Coupled time-space fractional mixed partial differential equations*

Consider the coupled time-space fractional mixed partial differential equation of the form:

$$\sum_{j=1}^{m_1} a_j\, {}_t\mathbb{T}^{j\alpha}_0 u(x,t) - \mu_1\, {}_t\mathbb{T}^{\alpha}_0 ({}_x\mathbb{T}^{\beta}_0 u(x,t)) =$$
$$F_1(x, u, v, {}_x\mathbb{T}^{\beta}_0 u(x,t), \ldots, {}_x\mathbb{T}^{l_1\beta}_0 u(x,t), {}_x\mathbb{T}^{l_2\beta}_0 v(x,t)) = F_1(u, v), \tag{3.10}$$

$$\sum_{k=1}^{m_2} c_k\, {}_t\mathbb{T}^{k\alpha}_0 v(x,t) - \mu_2\, {}_t\mathbb{T}^{\alpha}_0 ({}_x\mathbb{T}^{\beta}_0 v(x,t)) =$$
$$F_2(x, u, v, {}_x\mathbb{T}^{\beta}_0 u(x,t), \ldots, {}_x\mathbb{T}^{l_3\beta}_0 u(x,t), {}_x\mathbb{T}^{l_4\beta}_0 v(x,t)) = F_2(u, v), \tag{3.11}$$

where $\alpha \in (k_1 - 1, k_1]$ and $\beta \in (k_2 - 1, k_2]$, i.e. $\alpha \in \mathbb{Z}_\delta$ and $\beta \in \mathbb{Z}_\gamma$ for $\delta \in (0, 1]$, $\gamma \in (0, 1]$, $k_1, k_2, l_1, l_2, l_3, l_4 \in \mathbb{N}$; $\mu_1, \mu_2, a_j, c_k \in \mathbb{R}$. $F_1(u, v)$, $F_2(u, v)$ are linear/nonlinear operators of u, v and their fractional derivatives with respect to the independent variable x.

Theorem 3.2. *Let a finite dimensional vector space* $L_{n_1} \times L_{n_2} = span\{l_1^1(x), \ldots, l_{n_1}^1(x)\} \times span\{l_1^2(x), \ldots, l_{n_2}^2(x)\}$ *be the invariant subspace under the operators* $F_m(u, v)$, $m = 1, 2$. *Moreover,* L_{n_1} *and* L_{n_2} *are invariant subspaces under the operators* ${}_t\mathrm{T}_0^{\alpha}u(x,t)$ *and* ${}_t\mathrm{T}_0^{\beta}v(x,t)$*, respectively. Then, the coupled FPDEs (3.10), (3.11) has an exact solution as follows*

$$u(x,t) = \sum_{i=1}^{n_1} \lambda_i^1(t) l_i^1(x), \tag{3.12}$$

$$v(x,t) = \sum_{i=1}^{n_2} \lambda_i^2(t) l_i^2(x), \tag{3.13}$$

where $\lambda_i^1(t)$ *satisfy a system of modified conformable fractional ODEs*

$$\sum_{j=1}^{m_1} a_j\, {}_t\mathrm{T}_0^{j\alpha}(\lambda_i^1(t)) - \mu_1\, {}_t\mathrm{T}_0^{\alpha}\phi_{n_1+i}(\lambda_1^1(t), \ldots, \lambda_{n_1}^1(t)) =$$
$$\phi_i^1(\lambda_1^1(t), \ldots, \lambda_{n_1}^1(t), \lambda_1^2(t), \ldots, \lambda_{n_2}^2(t)),\ i = 1, 2, \ldots, n_1 \tag{3.14}$$

and $\lambda_i^2(t)$ *satisfy a system of modified conformable fractional ODEs*

$$\sum_{k=1}^{m_2} c_k\, {}_t\mathrm{T}_0^{k\alpha}(\lambda_i^2(t)) - \mu_2\, {}_t\mathrm{T}_0^{\alpha}\phi_{n_2+i}(\lambda_1^2(t), \ldots, \lambda_{n_2}^2(t)) =$$
$$\phi_i^2(\lambda_1^1(t), \ldots, \lambda_{n_1}^1(t), \lambda_1^2(t), \ldots, \lambda_{n_2}^2(t)),\ i = 1, \ldots, n_2 \tag{3.15}$$

where $\phi_1^1, \ldots, \phi_{n_1}^1$ *are the expansion coefficients of* $F_1(u, v)$ *with respect to* $L_{n_1} \times L_{n_2}$, $\phi_{n_1+1}^1, \ldots, \phi_{2n_1}^1$ *are the expansion coefficients of* ${}_t\mathrm{T}_0^{\beta}u(x,t)$ *with respect to* L_{n_1}, $\phi_1^2, \ldots, \phi_{n_2}^2$ *are the expansion coefficients of* $F_2(u, v)$ *with respect to* $L_{n_1} \times L_{n_2}$, $\phi_{n_2+1}^2, \ldots, \phi_{2n_2}^2$ *are the expansion coefficients of* ${}_t\mathrm{T}_0^{\beta}v(x,t)$ *with respect to* L_{n_2}.

Proof. From $u(x,t) = \sum_{i=1}^{n_1} \lambda_i^1(t) l_i^1(x)$, $v(x,t) = \sum_{i=1}^{n_2} \lambda_i^2(t) l_i^2(x)$ and linearity of modified conformable fractional derivative, we obtain

$$\sum_{j=1}^{m_1} a_j\, {}_t\mathrm{T}_0^{j\alpha}u(x,t) = \sum_{j=1}^{m_1} a_j\, {}_t\mathrm{T}_0^{j\alpha}\Big(\sum_{i=1}^{n_1} \lambda_i^1(t) l_i^1(x)\Big) = \sum_{i=1}^{n_1}\Big(\sum_{j=1}^{m_1} a_j\, {}_t\mathrm{T}_0^{j\alpha}\lambda_i^1(t)\Big) l_i^1(x), \tag{3.16}$$

$$\sum_{k=1}^{m_2} c_k\, {}_t\mathrm{T}_0^{k\alpha}v(x,t) = \sum_{k=1}^{m_2} c_k\, {}_t\mathrm{T}_0^{k\alpha}\Big(\sum_{i=1}^{n_2} \lambda_i^2(t) l_i^2(x)\Big) = \sum_{i=1}^{n_2}\Big(\sum_{k=1}^{m_2} c_k\, {}_t\mathrm{T}_0^{k\alpha}\lambda_i^2(t)\Big) l_i^2(x) \tag{3.17}$$

since $L_{n_1} \times L_{n_2} = span\{l_1^1(x), \ldots, l_{n_1}^1(x)\} \times span\{l_1^2(x), \ldots, l_{n_2}^2(x)\}$ is an invariant subspace under the operators $F_m(u, v)$, $m = 1, 2$ and L_{n_1}, L_{n_2} are invariant under ${}_t\mathrm{T}_0^{\beta}u(x,t)$, ${}_t\mathrm{T}_0^{\beta}v(x,t)$, respectively, there exit expansion coefficient functions such that

$$F_m(u, v) = \sum_{i=1}^{n_m} \phi_i^m(\lambda_1^1(t), \ldots, \lambda_{n_1}^1(t), \lambda_1^2(t), \ldots, \lambda_{n_2}^2(t))\, l_i^m(x), m = 1, 2, \tag{3.18}$$

$${}_t\mathrm{T}_0^{\beta}\Big(\sum_{i=1}^{n_1} \lambda_i^1(t) l_i^1(x)\Big) = \sum_{i=1}^{n_1} \phi_{n_1+i}^1(\lambda_1^1(t), \ldots, \lambda_{n_1}^1(t)) l_i^1(x), \tag{3.19}$$

$${}_t\mathrm{T}_0^{\beta}\Big(\sum_{i=1}^{n_2} \lambda_i^2(t) l_i^2(x)\Big) = \sum_{i=1}^{n_2} \phi_{n_2+i}^2(\lambda_1^2(t), \ldots, \lambda_{n_2}^2(t)) l_i^2(x), \tag{3.20}$$

where $\phi_1^1, \ldots, \phi_{n_1}^1$ are the expansion coefficients of $F_1(u, v)$ with respect to $L_{n_1} \times L_{n_2}$, $\phi_1^2, \ldots, \phi_{n_2}^2$ are the expansion coefficients of $F_2(u, v)$ with respect to $L_{n_1} \times L_{n_2}$, $\phi_{n_1+1}^1, \ldots, \phi_{2n_1}^1$ are the expansion coefficients of ${}_t\mathrm{T}_0^{\beta}u$ with respect to L_{n_1}, $\phi_{n_2+1}^1, \ldots, \phi_{2n_2}^1$ are the expansion coefficients of ${}_t\mathrm{T}_0^{\beta}v$ with respect to L_{n_2}.

From Eqs. (3.18), (3.19) and (3.20)

$$F_1(u, v) + \mu_1\, {}_t\mathrm{T}_{0+}^{\alpha}\mathrm{T}_0^{\beta}u(x,t) = \sum_{i=1}^{n_1} \phi_i^1(\lambda_1^1(t), \ldots, \lambda_{n_1}^1(t), \lambda_1^2(t), \ldots, \lambda_{n_2}^2(t))\, l_i^1(x)$$
$$+ \mu_1\, {}_t\mathrm{T}_0^{\alpha}\phi_{n_1+i}^1(\lambda_1^1(t), \ldots, \lambda_{n_1}^1(t)) l_i^1(x), \tag{3.21}$$

$$F_2(u, v) + \mu_2\, {}_t\mathrm{T}_{0+}^{\alpha}\mathrm{T}_0^{\beta}v(x,t) = \sum_{i=1}^{n_2} \phi_i^2(\lambda_1^1(t), \ldots, \lambda_{n_1}^1(t), \lambda_1^2(t), \ldots, \lambda_{n_2}^2(t))$$
$$+ \mu_2\, {}_t\mathrm{T}_0^{\alpha}\phi_{n_2+i}^2(\lambda_1^2(t), \ldots, \lambda_{n_2}^2(t)) l_i^2(x) \tag{3.22}$$

Eqs. (3.16) and (3.21) are substituted into Eq. (3.10) to obtain

$$\sum_{i=1}^{n_1}\Big(\sum_{j=1}^{m_1} a_j\, {}_t\mathrm{T}_0^{j\alpha}\lambda_i^1(t) - \phi_i^1(\lambda_1^1(t), \ldots, \lambda_{n_1}^1(t), \lambda_1^2(t), \ldots, \lambda_{n_2}^2(t))$$
$$- \mu_1\, {}_t\mathrm{T}_0^{\alpha}\phi_{n_1+i}^1(\lambda_1^1(t), \ldots, \lambda_{n_1}^1(t))\Big) l_i^1(x) = 0. \tag{3.23}$$

Eqs. (3.17) and (3.22) are substituted into Eq. (3.11) to obtain

$$\sum_{i=1}^{n_2}\Big(\sum_{k=1}^{m_2} c_k\, {}_t\mathrm{T}_0^{k\alpha}\lambda_i^2(t) - \phi_i^2(\lambda_1^1(t), \ldots, \lambda_{n_1}^1(t), \lambda_1^2(t), \ldots, \lambda_{n_2}^2(t))$$
$$- \mu_2\, {}_t\mathrm{T}_0^{\alpha}\phi_{n_2+i}^2(\lambda_1^2(t), \ldots, \lambda_{n_2}^2(t))\Big) l_i^2(x) = 0. \tag{3.24}$$

Using Eqs. (3.23) and (3.24) and the fact that $\{l_1^1(x), \ldots, l_{n_1}^1(x)\}$, $\{l_1^2(x), \ldots, l_{n_2}^2(x)\}$ are linearly independent set, we have the system of FDEs as follows:

$$\sum_{j=1}^{m_1} a_j\, {}_t\mathrm{T}_0^{j\alpha}\lambda_i^1(t) - \mu_1\, {}_t\mathrm{T}_0^{\alpha}\phi_{n_1+i}^1(\lambda_1^1(t), \ldots, \lambda_{n_1}^1(t)) =$$
$$\phi_i^1(\lambda_1^1(t), \ldots, \lambda_{n_1}^1(t), \lambda_1^2(t), \ldots, \lambda_{n_2}^2(t)),\ i = 1, \ldots, n_1 \tag{3.25}$$

$$\sum_{k=1}^{m_2} c_k\, {}_t\mathrm{T}_0^{k\alpha}\lambda_i^2(t) - \mu_2\, {}_t\mathrm{T}_0^{\alpha}\phi_{n_2+i}^2(\lambda_1^2(t), \ldots, \lambda_{n_2}^2(t)) =$$
$$\phi_i^2(\lambda_1^1(t), \ldots, \lambda_{n_1}^1(t), \lambda_1^2(t), \ldots, \lambda_{n_2}^2(t)),\ i = 1, \ldots, n_2 \tag{3.26}$$

Hence proved.

4. Applications

In this section, we give several examples to illustrate Theorems 3.1 and 3.2. The solution graphs are also shown for all the considered problems (see Fig. 1).

Example 4.1. Consider the following fractional generalization of K–dV equation with initial condition:

$${}_t\mathrm{T}^{\alpha}u(x,t) = {}_x\mathrm{T}^{\beta}({}_x\mathrm{T}^{\beta}({}_x\mathrm{T}^{\beta}(\tfrac{u^2}{2}))) + {}_t\mathrm{T}^{\alpha}({}_x\mathrm{T}^{\beta}u(x,t)),$$
$$u(x,0) = 1 + 0.1x^{\beta} + 0.01x^{2\beta}, \tag{4.1}$$

where $\alpha \in (0,1]$, $\beta \in (1,2]$ i.e. $\beta \in \chi_{\gamma}$ for some $\gamma \in (0,1]$.

Note that $L_3 = span\{1, x^{\beta}, x^{2\beta}\}$ is an invariant subspace under ${}_x\mathrm{T}^{\beta}u(x,t)$ and $F(u) = {}_x\mathrm{T}^{\beta}({}_x\mathrm{T}^{\beta}({}_x\mathrm{T}^{\beta}(\tfrac{u^2}{2})))$ as,

$${}_x\mathrm{T}^{\beta}(b_0(t) + b_1(t)x^{\beta} + b_2(t)x^{2\beta}) = b_1(t)\beta(\beta-\gamma) + b_2(t)2\beta(2\beta-\gamma)x^{\beta} \in L_3,$$

$$F(b_0(t) + b_1(t)x^{\beta} + b_2(t)x^{2\beta})$$
$$= 6\beta^3(\beta-\gamma)(2\beta-\gamma)(3\beta-\gamma)b_1(t)b_2(t)$$
$$+ 12\beta^3(2\beta-\gamma)(3\beta-\gamma)(4\beta-\gamma)b_2(t)^2x^{\beta} \in L_3.$$

It follows from the Theorem 3.1 that the above fractional generalization of K–dV equation has an exact solution of the form

$$u(x,t) = b_0(t) + b_1(t)x^{\beta} + b_2(t)x^{2\beta}, \tag{4.2}$$

such that $b_0(t)$, $b_1(t)$ and $b_2(t)$ satisfy the system of MCFDEs as follows:

$${}_t\mathrm{T}^{\alpha}b_0(t) - \beta(\beta-\gamma){}_t\mathrm{T}^{\alpha}b_1(t) = C_2b_1(t)b_2(t), \tag{4.3}$$

$${}_t\mathrm{T}^{\alpha}b_1(t) - 2\beta(2\beta-\gamma){}_t\mathrm{T}^{\alpha}b_2(t) = C_1b_2(t)^2, \tag{4.4}$$

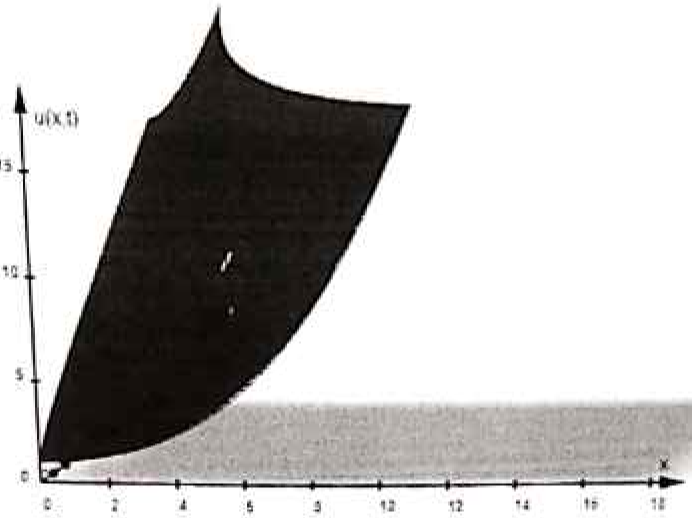

Fig. 1. The graph of the solution of Eq. (4.1) with $\alpha = 0.5$ and $\beta = 1.51$.

$$ {}_tT^{\alpha} b_2(t) = 0. \tag{4.5}$$

where $C_1 = 12\beta^3(2\beta-\gamma)(3\beta-\gamma)(4\beta-\gamma)$, $C_2 = 6\beta^3(\beta-\gamma)(2\beta-\gamma)(3\beta-\gamma)$.

Solving the above system of MCFDEs, we obtain $b_2(t) = D$, where D is a real constant. Hence Eqs. (4.3) and (4.4) have the form

$$ {}_tT^{\alpha} b_0(t) - \beta(\beta-\gamma)\,{}_tT^{\alpha} b_1(t) = C_2 D b_1(t), \tag{4.6}$$

$$ {}_tT^{\alpha} b_1(t) = C_1 D^2. \tag{4.7}$$

Solving Eq. (4.7), we obtain $b_1(t) = b_1(0) + C_1 D^2 \frac{t^{\alpha}}{\alpha}$. Substituting value of $b_1(t)$ in Eq. (4.6) we get,

$b_0(t) = b_0(0) + (\beta(\beta-\gamma)C_1 D^2 + C_2 D b_1(0))\frac{t^{\alpha}}{\alpha} + C_1 C_2 D^3 \frac{t^{2\alpha}}{2\alpha^2}$.

Then, an exact solution of the generalization of fractional K–dV equation is given by

$$u(x,t) = b_0(0) + (\beta(\beta-\gamma)C_1 D^2 + C_2 D b_1(0))\frac{t^{\alpha}}{\alpha} + C_1 C_2 D^3 \frac{t^{2\alpha}}{2\alpha^2}$$
$$+ b_1(0)x^{\beta} + C_1 D^2 \frac{t^{\alpha}}{\alpha} x^{\beta} + D x^{2\beta}.$$

Thus the exact solution of (4.1) along with the initial condition $u(x,0) = 1 + 0.1x^{\beta} + 0.01x^{2\beta}$ turns out to be

$$u(x,t) = 1 + (0.0001C_1\beta(\beta-\gamma) + 0.001C_2)\frac{t^{\alpha}}{\alpha} + 0.000001C_1C_2\frac{t^{2\alpha}}{2\alpha^2} +$$
$$0.1x^{\beta} + 0.0001C_1\frac{t^{\alpha}}{\alpha}x^{\beta} + 0.01x^{2\beta}.$$

Example 4.2. Consider the following fractional version of coupled Boussinesq equations with initial conditions:

$$\begin{aligned}
&{}_tT^{\alpha_1} u(x,t) + {}_tT^{\alpha_1}({}_xT^{\beta} u(x,t)) = -{}_xT^{\beta} v(x,t),\\
&{}_tT^{\alpha_2} v(x,t) + {}_tT^{\alpha_2}({}_xT^{\beta} v(x,t)) = -m_1\,{}_xT^{\beta} u(x,t) + 3u\,{}_xT^{\beta} u(x,t)\\
&\qquad + m_2\,{}_xT^{\beta}({}_xT^{\beta}({}_xT^{\beta} u(x,t))),\\
&u(x,0) = 2 + x^{\beta},\\
&v(x,0) = 1 + 6x^{\beta},
\end{aligned} \tag{4.8}$$

where $t > 0$, $0 < \alpha_1, \alpha_2, \beta \le 1$, m_1, m_2 are arbitrary constants, $F_1(u,v) = -{}_xT^{\beta} v(x,t)$, $F_2(u,v) = -m_1\,{}_xT^{\beta} u(x,t) + 3u\,{}_xT^{\beta} u(x,t) + m_2\,{}_xT^{\beta}({}_xT^{\beta}({}_xT^{\beta} u(x,t)))$.

Since $0 < \alpha_1, \alpha_2, \beta \le 1$, the modified conformable fractional derivative and conformable fractional derivative coincides (see Figs. 2–4).

Note that $L^2 = L_1^2 \times L_2^2 = \text{span}\{1, x^{\beta}\} \times \text{span}\{1, x^{\beta}\}$ is invariant under $F_1(u,v)$,

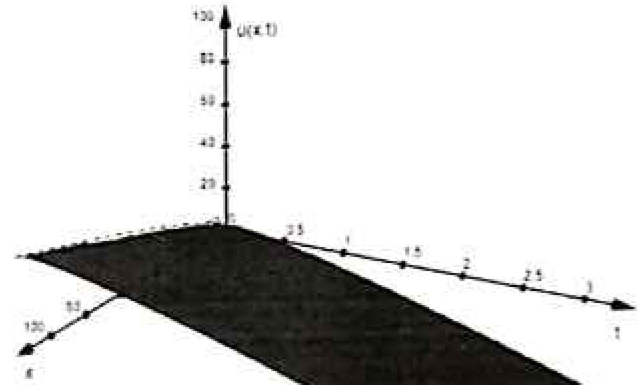

Fig. 2. The graph of the solution $u(x,t)$ of Eq. (4.8) with $\alpha_1 = 0.6$, $\alpha_2 = 0.5$ and $\beta = 0.8$.

$F_2(u,v)$ and L_1^2, L_2^2 are invariant under ${}_xT^{\beta}u(x,t)$, ${}_xT^{\beta}v(x,t)$, respectively, as

${}_xT^{\beta}(a_0(t) + a_1(t)x^{\beta}) = a_1(t)\beta \in L_1^2$ and ${}_xT^{\beta}(b_0(t) + b_1(t)x^{\beta}) = b_1(t)\beta \in L_2^2$,

$F_1(a_0(t) + a_1(t)x^{\beta}, b_0(t) + b_1(t)x^{\beta}) = -b_1(t)\beta \in L_1^2$,

$F_2(a_0(t) + a_1(t)x^{\beta}, b_0(t) + b_1(t)x^{\beta}) = -m_1 a_1(t)\beta + 3a_0(t)a_1(t)\beta + 3a_1^2(t)\beta x^{\beta} \in L_2^2$.

It follows from the Theorem 3.2 that the above fractional generalization of coupled Boussinesq equation has an exact solution of the form

$$u(x,t) = a_0(t) + a_1(t)x^{\beta}, \tag{4.9}$$

$$v(x,t) = b_0(t) + b_1(t)x^{\beta}, \tag{4.10}$$

such that $a_0(t), a_1(t), b_0(t)$ and $b_1(t)$ satisfy the system of MCFDEs:

$$\begin{aligned}
&{}_tT^{\alpha_1} a_0(t) - \beta\,{}_tT^{\alpha_1} a_1(t) = -\beta b_1(t),\\
&{}_tT^{\alpha_1} a_1(t) = 0,\\
&{}_tT^{\alpha_2} b_0(t) - {}_tT^{\alpha_2} b_1(t) = -m_1 a_1(t)\beta + 3a_0(t)a_1(t)\beta,\\
&{}_tT^{\alpha_2} b_1(t) = 3\beta a_1^2(t).
\end{aligned}$$

After solving the above system of MCFDEs, we obtain

$a_1(t) = D$, where D is a real constant,

$b_1(t) = b_1(0) + \frac{3D^2\beta t^{\alpha_2}}{\alpha_2}$,

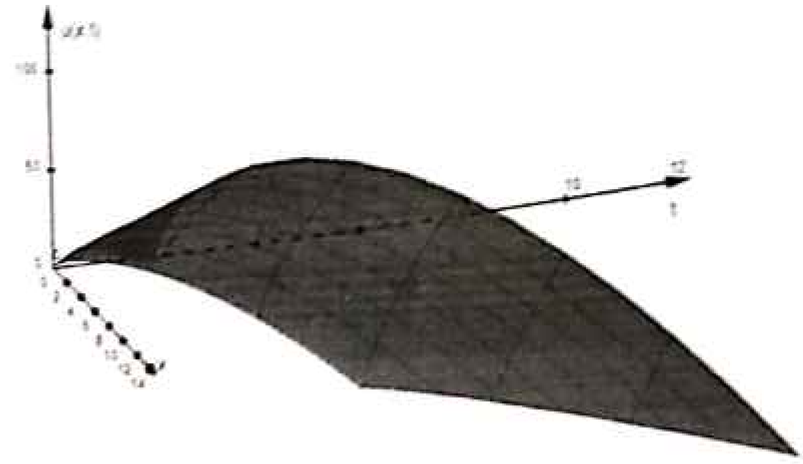

Fig. 3. The graph of the solution $u(x,t)$ of Eq. (4.8) with $\alpha_1 = 0.5$, $\alpha_2 = 0.5$ and $\beta = 0.8$.

$$a_0(t) = a_0(0) - \frac{\beta b_1(0)t^{\alpha_1}}{\alpha_1} - \frac{3\beta^2 D^2 t^{\alpha_1+\alpha_2}}{\alpha_1\alpha_2+\alpha_2^2},$$

$$b_0(t) = b_0(0) + \frac{(-3\beta^2 D^2 - m_1\beta D + 3\beta D a_0(0))t^{\alpha_2}}{\alpha_2} - \frac{3\beta^2 D b_1(0)t^{\alpha_1+\alpha_2}}{\alpha_1^2+\alpha_1\alpha_2} - \frac{9\beta^3 D^3 t^{\alpha_1+2\alpha_2}}{\alpha_1^2\alpha_2+3\alpha_2^2\alpha_1+2\alpha_2^3}.$$

Now, from Eqs. (4.9) and (4.10), exact solution of the generalization of coupled Boussinesq equation is given by

$$u(x,t) = a_0(0) - \frac{\beta b_1(0)t^{\alpha_1}}{\alpha_1} - \frac{3\beta^2 D^2 t^{\alpha_1+\alpha_2}}{\alpha_1\alpha_2+\alpha_2^2} + Dx^\beta,$$

$$v(x,t) = b_0(0) + \frac{(-3\beta^2 D^2 - m_1\beta D + 3\beta D a_0(0))t^{\alpha_2}}{\alpha_2} - \frac{3\beta^2 D b_1(0)t^{\alpha_1+\alpha_2}}{\alpha_1^2+\alpha_1\alpha_2}$$
$$- \frac{9\beta^3 D^3 t^{\alpha_1+2\alpha_2}}{\alpha_1^2\alpha_2+3\alpha_2^2\alpha_1+2\alpha_2^3} + b_1(0)x^\beta + \frac{3D^2\beta t^{\alpha_2}}{\alpha_2}x^\beta.$$

Thus the exact solution of (4.8) along with the initial conditions $u(x,0) = 2 + 1x^\beta$, $v(x,0) = 1 + 6x^\beta$ turns out to be

$$u(x,t) = 6 - \frac{6\beta t^{\alpha_1}}{\alpha_1} - \frac{18\beta^2 t^{\alpha_1+\alpha_2}}{\alpha_1\alpha_2+\alpha_2^2} + x^\beta,$$

$$v(x,t) = 1 + \frac{(-3\beta^2 - m_1\beta + 6\beta)t^{\alpha_2}}{\alpha_2} - \frac{18\beta^2 t^{\alpha_1+\alpha_2}}{\alpha_1^2+\alpha_1\alpha_2}$$
$$- \frac{9\beta^3 t^{\alpha_1+2\alpha_2}}{\alpha_1^2\alpha_2+3\alpha_2^2\alpha_1+2\alpha_2^3} + 6x^\beta + \frac{3\beta t^{\alpha_2}}{\alpha_2}x^\beta$$

Example 4.3. Consider the following fractional generalization of telegraph equation:

$$_tT^{2\alpha}u(x,t) + 2\,_tT^{\alpha}u(x,t) = \,_xT^{\beta}u(x,t) - u. \tag{4.11}$$

where $\alpha \in (0,1]$, $\beta \in (1,2]$ i.e. $\beta \in \mathcal{I}_\gamma$ for some $\gamma \in (0,1]$.

Since $\beta \in \mathcal{I}_{\gamma=\beta}$, it implies that $_xT^{\beta}u(x,t) = \,_xT^{2\gamma}u(x,t)$. Note that $L_2 = span\{1, exp(\frac{x^\gamma}{\gamma})\}$ is invariant under $F(u) = \,_xT^{\beta}u(x,t) - u(x,t)$ as

$$F(b_0(t) + b_1(t)exp(\frac{x^\gamma}{\gamma})) = \,_xT^{2\gamma}(b_0(t) + b_1(t)exp(\frac{x^\gamma}{\gamma})) - b_0(t) - b_1(t)exp(\frac{x^\gamma}{\gamma})$$
$$= -b_0 \in L_2.$$

It follows from the Theorem 3.1 that the above fractional generalization of telegraph equation has an exact solution of the form

$$u(x,t) = b_0(t) + b_1(t)exp(\frac{x^\gamma}{\gamma}), \tag{4.12}$$

such that $b_0(t)$ and $b_1(t)$ satisfy the system of MCFDEs as follows:

$$_tT^{2\alpha}b_0(t) + 2\,_tT^{\alpha}b_0(t) = -b_0(t). \tag{4.13}$$

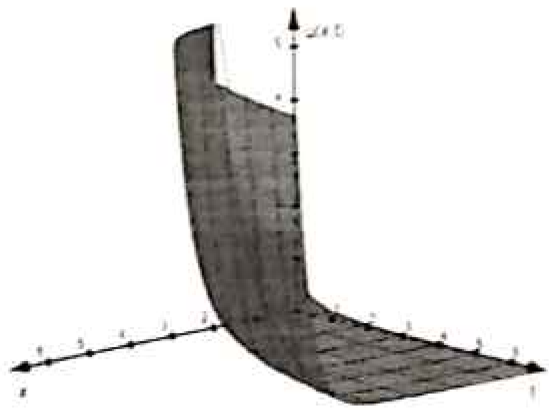

Fig. 4. The graph of the solution of Eq. (4.11) with $\alpha = 0.5$ and $\beta = 1.75$.

$$_tT^{2\alpha}b_1(t) + 2\,_tT^{\alpha}b_1(t) = 0. \tag{4.14}$$

Solving the above MCFDEs (4.13) and (4.14), we obtain

$b_0(t) = exp(-\frac{t^\alpha}{\alpha})$ and $b_1(t) = exp(-\frac{2t^\alpha}{\alpha})$.

Substituting the values of $b_0(t)$ and $b_1(t)$ in Eq. (4.12), we obtain an exact solution of telegraph equation as follows

$$u(x,t) = exp(-\frac{t^\alpha}{\alpha}) + exp(-\frac{2t^\alpha}{\alpha})exp(\frac{x^\gamma}{\gamma}).$$

Example 4.4. Consider the following time–space mixed fractional nonlinear water wave equation (NWWE):

$$_tT^{\alpha}u(x,t) = -\,_xT^{\beta}u(x,t) - c_1\,u(x,t)\,_xT^{\beta}u(x,t) - c_2\,_xT^{\beta}(_xT^{\beta}(_xT^{\beta}u(x,t)))$$
$$- c_3\,_xT^{\beta}u(x,t)\,_xT^{\beta}(_xT^{\beta}u(x,t)) - c_4\,u(x,t)\,_xT^{\beta}(_xT^{\beta}(_xT^{\beta}u(x,t)))$$
$$- c_5\,_xT^{\beta}(_xT^{\beta}(_xT^{\beta}(_xT^{\beta}(_xT^{\beta}u(x,t))))) = F(u) - \,_tT^{\alpha}(_xT^{\beta}u(x,t)),$$

where $\alpha \in (0,1]$, $\beta \in (0,1]$ and c_1, c_2, c_3, c_4, c_5 are real constant.

Since $0 < \alpha, \beta \leq 1$, the modified conformable fractional derivative and conformable fractional derivative coincides.

Note that I_2=span$\{1, x^\beta\}$ is invariant under $F(u)$ and $_xT^{\beta}u(x,t)$ as

$$F(b_0(t) + b_1(t)x^\beta) = -b_1(t)\beta - c_1 b_0(t)b_1(t)\beta - c_1 b_1^2(t)\beta x^\beta. \tag{4.15}$$

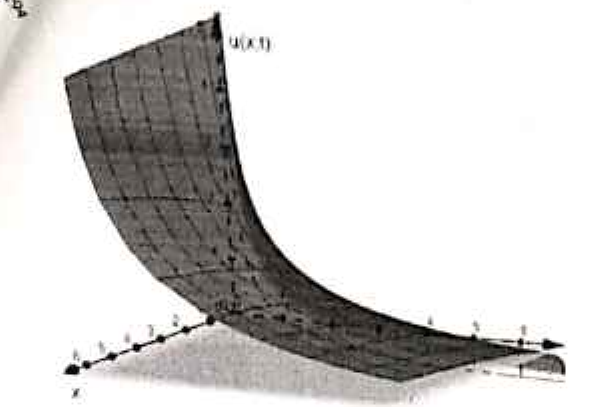

Fig. 5. The graph of the solution of time-space fractional mixed NWWE with $c=1$, $c_1=1$, $\alpha=0.5$ and $\beta=0.25$.

$$ {}_t\mathbb{T}^{\beta}(b_0(t)+b_1(t)x^{\beta}) = b_1(t)\beta. \tag{4.16}$$

It follows from the Theorem 3.1 that the above time–space fractional nonlinear water wave equation has an exact solution of the form

$$u(x,t) = b_0(t) + b_1(t)x^{\beta}, \tag{4.17}$$

such that $b_0(t)$ and $b_1(t)$ satisfy the following system of MCFDEs

$$ {}_t\mathbb{T}^{\alpha} b_0(t) - \beta_t\mathbb{T}^{\alpha} b_1(t) = -b_1(t)\beta - c_1 b_0(t)b_1(t)\beta, \tag{4.18}$$

$$ {}_t\mathbb{T}^{\alpha} b_1(t) = -c_1 b_1^2(t)\beta. \tag{4.19}$$

Solving the above MCFDEs (4.18) and (4.19), we obtain

$b_0(t) = ct^{-\alpha} - \alpha^2 c_1^{-1} t^{-\alpha}\log(t) - c_1^{-1}$, where c is any real constant and $b_1(t) = \alpha\beta^{-1}c_1^{-1}t^{-\alpha}$ (see Fig. 5).

Substituting the values of $b_0(t)$ and $b_1(t)$ in Eq. (4.17), we obtain an exact solution of the time–space fractional mixed nonlinear water wave equation as follows

$$u(x,t) = ct^{-\alpha} - \alpha^2 c_1^{-1} t^{-\alpha}\log(t) - c_1^{-1} + \alpha\beta^{-1}c_1^{-1}t^{-\alpha}x^{\beta}.$$

Next, we will present the comparison of exact solutions to FPDEs with MCFD and Ca-FD. For this, we consider some examples of time–space fractional partial differential equations.

Burgers' equation with Ca-FD. Consider the time–space fractional Burgers' equation with Caputo fractional derivative:

$$ {}_tD^{\alpha}u(x,t) = -u(x,t)_xD^{\beta}u(x,t) + \mu_xD^{\beta}({}_xD^{\beta}u(x,t)) = F(u), \tag{4.20}$$

where $\alpha \in (0,1)\setminus\{\frac{1}{2}\}$, $\beta \in (0,1]$ and μ is a diffusion constant.

Note that L_2=span$\{1, x^{\beta}\}$ is invariant under $F(u)$ as

$$F(b_0(t) + b_1(t)x^{\beta}) = -b_0(t)b_1(t)\Gamma(1+\beta) - b_1^2(t)\Gamma(1+\beta)x^{\beta} \in L_2.$$

It follows from Ref. 20 that the above time–space fractional Burgers' equation has an exact solution of the form

$$u(x,t) = b_0(t) + b_1(t)x^{\beta}, \tag{4.21}$$

such that $b_0(t)$ and $b_1(t)$ satisfy the following system of Ca-FDEs

$$ {}_tD^{\alpha} b_0(t) = -b_0(t)b_1(t)\Gamma(1+\beta),$$

$$ {}_tD^{\alpha} b_1(t) = -b_1^2(t)\Gamma(1+\beta)x^{\beta}.$$

Solving the above Ca-FDEs, we obtain

$b_1(t) = -\frac{\Gamma(1-\alpha)t^{-\alpha}}{\Gamma(1+\beta)\Gamma(1-2\alpha)}$ and $b_0(t) = Dt^{-\alpha}$, where D is a real constant.

Substituting the values of $b_0(t)$ and $b_1(t)$ in Eq. (4.21), we obtain an exact solution of the time–space fractional Burgers' equation as follows

$$u(x,t) = Dt^{-\alpha} - \frac{\Gamma(1-\alpha)t^{-\alpha}}{\Gamma(1+\beta)\Gamma(1-2\alpha)}x^{\beta}.$$

Burgers' equation with MCFD. Consider the time–space fractional Burgers' equation with modified conformable fractional derivative(MCFD):

$$ {}_t\mathbb{T}^{\alpha}u(x,t) = -u(x,t)_x\mathbb{T}^{\beta}u(x,t) + \mu_x\mathbb{T}^{\beta}({}_x\mathbb{T}^{\beta}u(x,t)) = F(u), \tag{4.22}$$

where $\alpha \in (0,1]$, $\beta \in (0,1]$ and μ is a diffusion constant.

Note that L_2=span$\{1, x^{\beta}\}$ is invariant under $F(u)$ as

$$F(b_0(t) + b_1(t)x^{\beta}) = -b_0(t)b_1(t)\beta - b_1^2(t)\beta x^{\beta} \in L_2.$$

It follows from the Theorem 3.1 that the above time–space fractional Burgers' equation has an exact solution of the form

$$u(x,t) = b_0(t) + b_1(t)x^{\beta}, \tag{4.23}$$

such that $b_0(t)$ and $b_1(t)$ satisfy the following system of MCFDEs

$$ {}_t\mathbb{T}^{\alpha} b_0(t) = -b_0(t)b_1(t)\beta,$$

$$ {}_t\mathbb{T}^{\alpha} b_1(t) = -b_1^2(t)\beta.$$

Solving the above MCFDEs, we obtain

$b_1(t) = \frac{\alpha t^{-\alpha}}{\beta}$ and $b_0(t) = Dt^{-\alpha}$, where D is a real constant.

Substituting the values of $b_0(t)$ and $b_1(t)$ in Eq. (4.23), we obtain an exact solution of the time–space fractional Burgers' equation as follows

$$u(x,t) = Dt^{-\alpha} + \frac{\alpha t^{-\alpha}}{\beta}x^{\beta}.$$

Figs. 6, 7 show the behaviour of the exact solution of Burgers' equation in the sense of Ca-FD and MCFD, respectively, for $\alpha = 0.8$, $\beta = 0.9$. Both the graphs show similar behaviour and smoothness property. Fig. 8 shows a good agreement between the exact solutions of Burgers' with Ca-FD and MCFD.

Fractional KdV equation with Ca-FD. Consider the time–space fractional Kdv equation with Caputo fractional derivative:

$$ {}_tD^{\alpha}u(x,t) = 6u(x,t)D^{\beta}u(x,t) + D^{2\beta}(D^{\beta}u(x,t)) = F(u), \tag{4.24}$$

where $\alpha \in (0,1)\setminus\{\frac{1}{2}\}$ and $\beta \in (0,1]$.

Note that L_2=span$\{1, x^{\beta}\}$ is invariant under $F(u)$ as

$$F(b_0(t) + b_1(t)x^{\beta}) = 6b_0(t)b_1(t)\Gamma(1+\beta) + 6b_1^2(t)\Gamma(1+\beta)x^{\beta} \in L_2.$$

It follows from Ref. 20 that the above time–space fractional KdV equation has an exact solution of the form

$$u(x,t) = b_0(t) + b_1(t)x^{\beta}, \tag{4.25}$$

such that $b_0(t)$ and $b_1(t)$ satisfy the following system of Ca-FDEs

$$ {}_tD^{\alpha} b_0(t) = 6b_0(t)b_1(t)\Gamma(1+\beta),$$

$$ {}_tD^{\alpha} b_1(t) = 6b_1^2(t)\Gamma(1+\beta)$$

Solving the above Ca-FDEs, we obtain

$b_0(t) = Dt^{-\alpha}$, where D is a real constant and $b_1(t) = \frac{\Gamma(1-\alpha)t^{-\alpha}}{6\Gamma(1+\beta)\Gamma(1-2\alpha)}$.

Substituting the values of $b_0(t)$ and $b_1(t)$ in Eq. (4.25), we obtain an exact solution of the time–space fractional KdV equation as follows

$$u(x,t) = Dt^{-\alpha} + \frac{\Gamma(1-\alpha)t^{-\alpha}}{6\Gamma(1+\beta)\Gamma(1-2\alpha)}x^{\beta}.$$

Fractional KdV equation with MCFD. Consider the time–space fractional Kdv equation with modified conformable fractional derivative:

$$ {}_t\mathbb{T}^{\alpha}u(x,t) = 6u(x,t)\mathbb{T}^{\beta}u(x,t) + \mathbb{T}^{2\beta}(\mathbb{T}^{\beta}u(x,t)) = F(u), \tag{4.26}$$

where $\alpha \in (0,1]$ and $\beta \in (0,1]$.

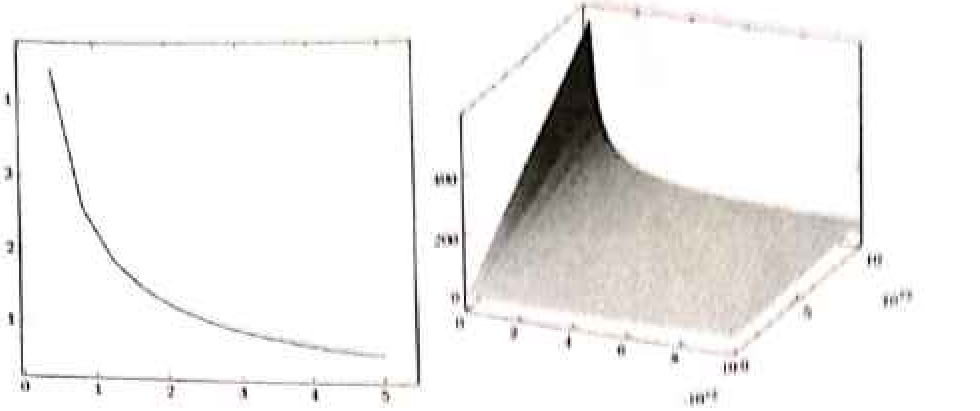

Fig. 6. 2D (with $x = 1$) and 3D graphs of the exact solution of Burgers' equation in the sense of Ca-FD with $D = 1$, $\alpha = 0.8$ and $\beta = 0.9$.

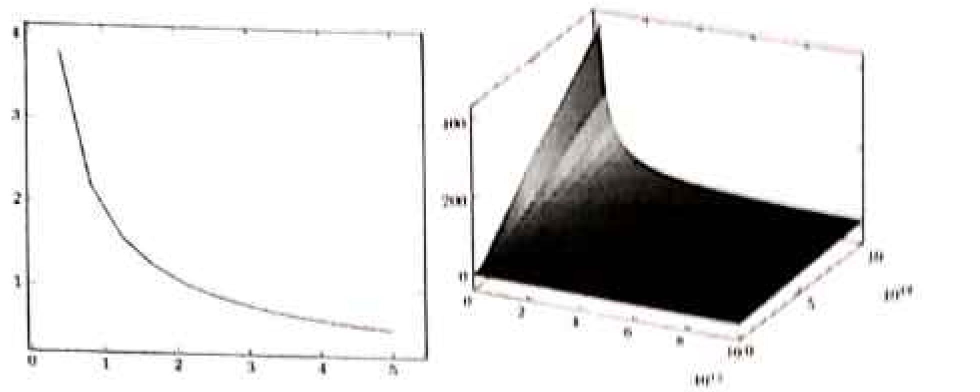

Fig. 7. 2D (with $x = 1$) and 3D graphs of the exact solution of Burgers' equation in the sense of MCFD with $D = 1$, $\alpha = 0.8$ and $\beta = 0.9$.

Fig. 8. Exact solution of Burgers' equation in the sense of MCFD (red) and Ca-FD (blue) with $\alpha = 0.8$, $\beta = 0.9$, $D = 1$ and $x = 1$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Note that L_2=span$\{1, x^\beta\}$ is invariant under $F(u)$ as

$$F(b_0(t) + b_1(t)x^\beta) = 6b_0(t)b_1(t)\beta + 6b_1^2(t)\beta x^\beta \in L_2.$$

It follows from the Theorem 3.1 that the above time-space fractional KdV equation has an exact solution of the form

$$u(x,t) = b_0(t) + b_1(t)x^\beta, \tag{4.27}$$

such that $b_0(t)$ and $b_1(t)$ satisfy the following system of MCFDEs

$${}_tT^\alpha b_0(t) = 6b_0(t)b_1(t)\beta,$$

$${}_tT^\alpha b_1(t) = 6b_1^2(t)\beta.$$

Solving the above MFDEs, we obtain

$b_0(t) = Dt^{-\alpha}$, where D is a real constant and $b_1(t) = -\frac{\alpha t^{-\alpha}}{6\beta}$.

Substituting the values of $b_0(t)$ and $b_1(t)$ in Eq. (4.27), we obtain an exact solution of the time-space fractional KdV equation as follows

$$u(x,t) = Dt^{-\alpha} - \frac{\alpha t^{-\alpha}}{6\beta}x^\beta.$$

Figs. 9, 10 show the behaviour of the exact solution of KdV equation in the sense of Ca-FD and MCFD, respectively, for $\alpha = 0.7$, $\beta = 0.6$. Both the graphs show similar behaviour and smoothness property. Fig. 11 shows a good agreement between the exact solutions of KdV with Ca-FD and MCFD.

KdV-B with Ca-FD. Consider the time-space fractional KdV-Burgers' (KdV-B) equation with Caputo fractional derivative:

$$\begin{aligned} {}_tD^\alpha u(x,t) = {} & \frac{\mu}{2}({}_xD^\beta u(x,t))^2 + c_1\,{}_xD^\beta({}_xD^\beta u(x,t)) \\ & - c_2\,{}_xD^\beta({}_xD^\beta({}_xD^\beta u(x,t))) = F(u), \end{aligned} \tag{4.28}$$

$$u(x,0) = 1 + 0.5x^\beta,$$

where $\alpha \in (0,1]$, $\beta \in (1,2]$ and μ, c_1, c_2 are real parameters.

Note that L_2=span$\{1, x^\beta\}$ is invariant under $F(u)$ as

$$F(b_0(t) + b_1(t)x^\beta) = \frac{\mu}{2}b_1^2(t)(\Gamma(1+\beta))^2.$$

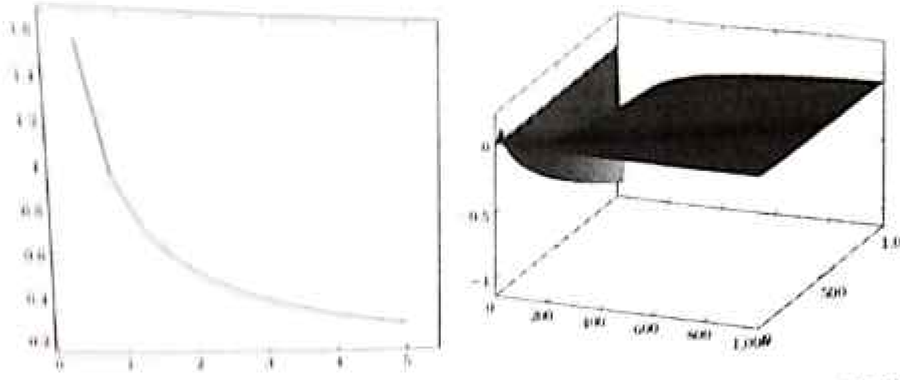

Fig. 9. 2D (with $x = 1$) and 3D graphs of the exact solution of KdV equation in the sense of Ca-FD with $D = 1$, $\alpha = 0.7$ and $\beta = 0.6$.

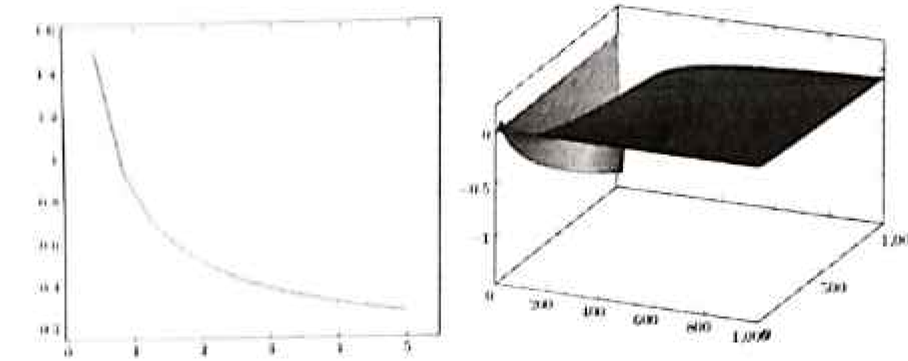

Fig. 10. 2D (with $x = 1$) and 3D graphs of the exact solution of KdV equation in the sense of MCFD with $D = 1$, $\alpha = 0.7$ and $\beta = 0.6$.

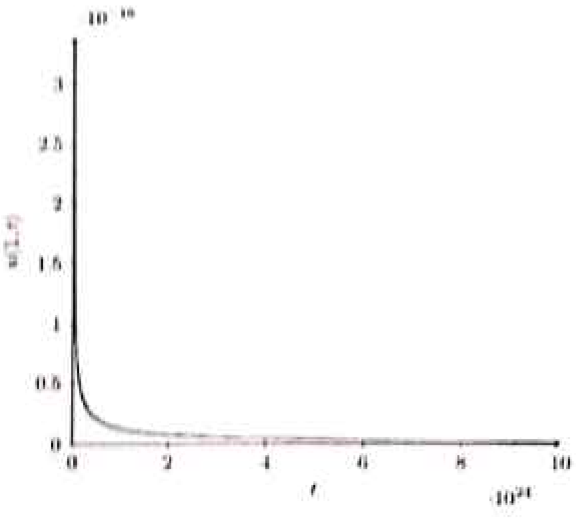

Fig. 11. Exact solution of KdV equation in the sense of MCFD (red) and Ca-FD (blue) with $\alpha = 0.7$, $\beta = 0.6$, $D = 1$ and $x = 1$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

It follows from Ref. 20 that the above time-space fractional KdV-Burgers' equation has an exact solution of the form

$$u(x,t) = b_0(t) + b_1(t)x^\beta, \tag{4.29}$$

such that $b_0(t)$ and $b_1(t)$ satisfy the following system of Ca-FDEs

$${}_tD^\alpha b_0(t) = \frac{\mu}{2} b_1^2(t)(\Gamma(1+\beta))^2,$$

$${}_tD^\alpha b_1(t) = 0$$

Solving the above Ca-FDEs, we obtain

$b_1(t) = D$, where D is a real constant and $b_0(t) = b_0(0) + \frac{\mu D^2 \Gamma(1+\beta)^2 t^\alpha}{2\Gamma(1+\alpha)}$. Substituting the values of $b_0(t)$ and $b_1(t)$ in Eq. (4.29), we obtain an exact solution of the time-space fractional KdV-Burgers' equation as follows

$$u(x,t) = b_0(0) + \frac{\mu D^2 \Gamma(1+\beta)^2 t^\alpha}{2\Gamma(1+\alpha)} + Dx^\beta.$$

Thus the exact solution of (4.28) along with the initial condition $u(x,0) = 1 + 0.5x^\beta$ turns out to be

$$u(x,t) = 1 + \frac{0.25\mu\Gamma(1+\beta)^2 t^\alpha}{2\Gamma(1+\alpha)} + 0.5x^\beta.$$

KdV-B with MCFD. Consider the time and space fractional KdV-Burgers' equation with modified conformable fractional derivative (MCFD):

$$\begin{aligned} {}_t\mathbb{T}^\alpha u(x,t) &= \frac{\mu}{2}({}_x\mathbb{T}^\beta u(x,t))^2 + c_1\, {}_x\mathbb{T}^\beta({}_x\mathbb{T}^\beta u(x,t)) \\ &\quad - c_2\, {}_x\mathbb{T}^\beta({}_x\mathbb{T}^\beta({}_x\mathbb{T}^\beta u(x,t))) = F(u), \\ u(x,0) &= 1 + 0.5x^\beta, \end{aligned} \tag{4.30}$$

where $\alpha \in (0,1]$, $\beta \in (1,2]$ i.e., $\beta \in \chi$, for some $\gamma \in (0,1]$ and μ, c_1, c_2 are real parameters.

Note that L_2=span$\{1, x^\beta\}$ is invariant under $F(u)$ as

$$F(b_0(t) + b_1(t)x^\beta) = \frac{\mu}{2} b_1^2(t)\beta^2(\beta-\gamma)^2$$

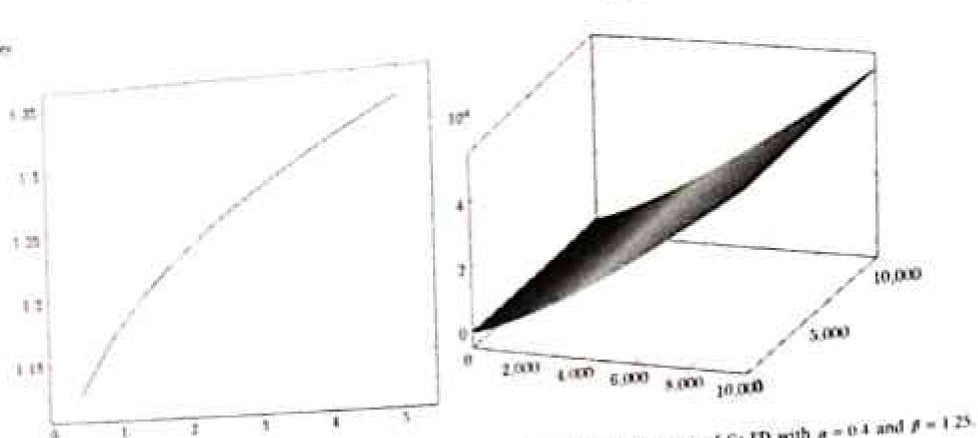

Fig. 12. 2D (with $x = 1$) and 3D graphs of the exact solution of KdV–Burgers' equation in the sense of Ca-FD with $\alpha = 0.4$ and $\beta = 1.25$.

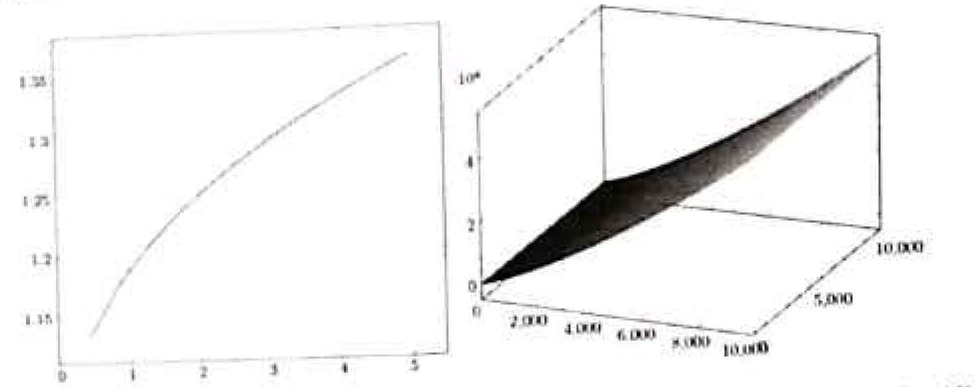

Fig. 13. 2D (with $x = 1$) and 3D graphs of the exact solution of KdV–Burgers' equation in the sense of MCFD with $\alpha = 0.4$ and $\beta = 1.25$.

It follows from the Theorem 3.1 that the above time–space fractional KdV–Burgers equation has an exact solution of the form

$$u(x,t) = h_0(t) + h_1(t)x^{\beta}, \tag{4.31}$$

such that $h_0(t)$ and $h_1(t)$ satisfy the following system of MCFDEs

$${}_tT^{\alpha} h_0(t) = \frac{\mu}{2} h_1^2(t)\beta^2(\beta-1)^2,$$

$${}_tT^{\alpha} h_1(t) = 0$$

Solving the above MCFDEs, we obtain

$h_1(t) = D$, where D is a real constant and $h_0(t) = h_0(0) + \frac{\mu D^2\beta^2(\beta-1)^2 t^{\alpha}}{2\alpha}$.

Substituting the values of $h_0(t)$ and $h_1(t)$ in Eq. (4.31), we obtain an exact solution of the time–space fractional KdV–Burgers equation as follows

$$u(x,t) = h_0(0) + \frac{\mu D^2\beta^2(\beta-1)^2 t^{\alpha}}{2\alpha} + Dx^{\beta}$$

Thus the exact solution of (4.30) along with the initial condition $u(x,0) = 1 + 0.5x^{\beta}$ turns out to be

$$u(x,t) = 1 + \frac{0.25\mu\beta^2(\beta-1)^2 t^{\alpha}}{2\alpha} + 0.5x^{\beta}$$

Figs. 12, 13 show the behaviour of the exact solution of KdV–Burgers' equation in the sense of Ca-FD and MCFD, respectively, for $\mu = 1$, $\alpha = 0.4$ and $\beta = 1.25$. Both the graphs show similar behaviour and smoothness property. Fig. 14 shows a good agreement between the exact solutions of KdV-B with Ca-FD and MCFD.

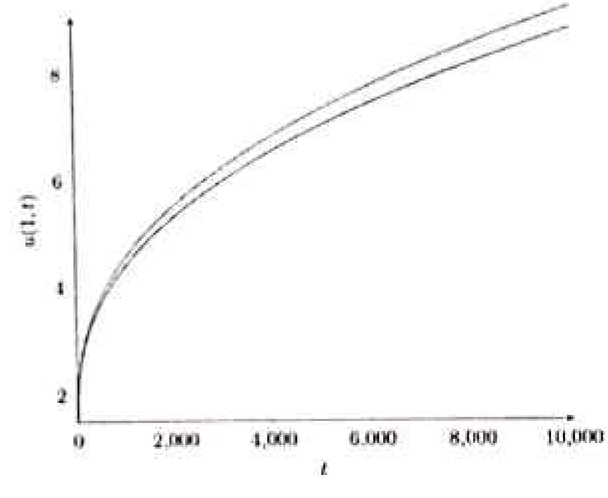

Fig. 14. Exact solution of KdV-Burgers' equation in the sense of MCFD (Red) and Ca-FD (Blue) with $\alpha = 0.4$, $\beta = 1.25$, $x = 1$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

5. Conclusion

Invariant subspace method is constructed for solving the linear/nonlinear FPDEs and the coupled FPDEs with fractional derivative in the sense of MCFD. Exact solution of some problems of FPDEs with certain initial conditions are obtained which shows the applicability of the method. Moreover, graphical comparison shows that MCFD is a good alternative of Caputo fractional derivative(Ca-FD) that may lead to many fruitful results in future.

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